IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Constructing solutions to the ultradiscrete Painlevé equations

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1997 J. Phys. A: Math. Gen. 30 7953

(http://iopscience.iop.org/0305-4470/30/22/029)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.110
The article was downloaded on 02/06/2010 at 06:05

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **30** (1997) 7953–7966. Printed in the UK PII: S0305-4470(97)84351-X

Constructing solutions to the ultradiscrete Painlevé equations

D Takahashi†, T Tokihiro‡, B Grammaticos§, Y Ohta‖ and A Ramani¶

† Department of Applied Mathematics and Informatics, Ryukoku University, Seta, Ohtsu 520-21, Japan

‡ Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan

§ LPN, Université Paris VII, Tour 24-14, 5ème étage, 75251 Paris, France

‖ Department of Applied Mathematics, Faculty of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739, Japan

¶ CPT, Ecole Polytechnique, CNRS, UPR 14, 91128 Palaiseau, France

Received 13 May 1997

Abstract. We investigate the nature of particular solutions to the ultradiscrete Painlevé equations. We start by analysing the autonomous limit and show that the equations possess an explicit invariant which leads naturally to the ultradiscrete analogue of elliptic functions. For the ultradiscrete Painlevé equations II and III we present special solutions reminiscent of the Casorati determinant ones which exist in the continuous and discrete cases. Finally we analyse the discrete Painlevé equation I and show how it contains both the continuous and the ultradiscrete ones as particular limits.

1. Introduction

The study of integrable cellular automata (CA) has received a substantial boost recently with the introduction of the ultradiscretization method which allows a *systematic* construction of CA's starting from a given discrete system [1]. At the heart of the method lies the transformation which relates the variable of the discrete system x to that of the ultradiscrete X, through $x = \mathrm{e}^{X/\epsilon}$. An essential requirement is that the variables of the discrete equation assume only *positive* values. In practice this means that the ultradiscretization will isolate the positive solutions of the discrete system. In order to obtain a CA one performs that limit $\epsilon \to +0$. The cornerstone of the procedure is the identity $\lim_{\epsilon\to+0} \epsilon \log(\mathrm{e}^{a/\epsilon} + \mathrm{e}^{b/\epsilon}) = \max(a, b)$. Thus, if a, b are integer, the result of the operation $\max(a, b)$ will give an integer value.

The ultradiscretization approach has made possible the systematic derivation of the CA analogues of a host of integrable evolution equations. In a recent work, we have presented the ultradiscrete forms of paradigmatic integrable systems: the Painlevé equations (Ps) [2]. Our approach was based on the ultradiscretization procedure applied to the known discrete forms of the Ps. In order to ensure the positivity requirement the discrete forms considered were the multiplicative ones, i.e. the q-Ps. Thus, for example, we start from the three expressions of q-Painlevé I equation (q-$\mathrm{P_I}$) [3]:

$$x_n^{\sigma} x_{n+1} x_{n-1} = z x_n + 1 \tag{1.1}$$

0305-4470/97/227953+14$19.50 © 1997 IOP Publishing Ltd

where $\sigma = 0, 1, 2$, $z = \lambda^n$. From (1.1), with $\lambda = e^{1/\epsilon}$, we obtain the ultradiscrete forms:

$$X_{n+1} + X_{n-1} + \sigma X_n = \max(0, X_n + n). \tag{1.2}$$

Ultradiscrete forms have been proposed for all the Ps. One important question that can be raised at this point is whether the ultradiscrete Ps (u-Ps) are indeed Ps. That latter have been proposed, initially in a continuous setting, as equations defining new transcendents, thus extending the special functions to the nonlinear domain. It is by now clear that the discrete Ps also fulfil the basic requirements and can be considered as defining new functions (of the appropriate discrete variable). However, the same depth of analysis is far from being reached for u-Ps. Thus, it is important that the properties of these new equations be studied in detail. In this paper we shall examine the particular solutions of the u-Ps II and III. In the discrete case, as well as the continuous one, the Ps are known to possess particular solutions for special values of their parameters. We shall show that the same holds true for the u-Ps and give the ultradiscrete analogue of the Casorati determinant-type solutions which have been established for the continuous and discrete Ps [4].

In section 2 we shall start with a simpler problem namely that of the autonomous limit of the u-Ps. In the continuous and discrete autonomous cases, the solutions are explicitly known to be elliptic functions. We study here the ultradiscrete equations and establish an integral of motion (which exists only in the autonomous limit). This integral would define the ultradiscrete analogue of the elliptic functions. However, in contrast to the continuous and discrete cases this does not seem possible. Thus we address directly the question of the ultradiscrete analogue of elliptic functions and show how the latter can be systematically constructed. Section 3 is devoted to the special solutions of u-P_{II} and u-P_{III}: these are the ultradiscrete analogues of the Casorati determinant rational solutions of the continuous and discrete Ps. Finally, we examine the ultradiscrete Painlevé I equation (u-P_I). The Painlevé I (P_I) equation does not have any particular solutions. Still, it is very interesting to study the discrete P_I equation and show how its solution contains the ones of the continuous as well as of the u-P_I. The transition from one to the other is mediated by the sign of the parameter. By changing this parameter we can move from a strictly positive, CA-like solution to the typical P_I solution with poles on the negative real axis.

2. Autonomous ultradiscrete equations and elliptic functions

Before proceeding to the study of special solutions of the u-Ps, let us start with a simpler example, that of their autonomous limits. It is well known that the autonomous limits of continuous and discrete Ps are solved in terms of elliptic functions [5]. In all these cases the second-order equation can be integrated once and it turns out that the resulting integral is just a known addition formula for elliptic functions. We can thus wonder whether these properties cross over to the ultradiscrete case.

Let us start with the multiplicative autonomous form of the discrete Painlevé I equation (d-P_I):

$$x_n^\sigma x_{n+1} x_{n-1} = \alpha x_n + 1. \tag{2.1}$$

It is easy to show that all these cases possess an invariant. Thus we have for $\sigma = 0$

$$x_n^2 + x_{n-1}^2 + \alpha(x_n + x_{n-1}) + 1 = k x_n x_{n-1} \tag{2.2a}$$

for $\sigma = 1$

$$x_n^2 x_{n-1}^2 + \alpha x_n x_{n-1}(x_n + x_{n-1}) + (\alpha^2 + 1)(x_n + x_{n-1}) + \alpha = k x_n x_{n-1} \tag{2.2b}$$

for $\sigma = 2$

$$x_n^2 x_{n-1}^2 + \alpha(x_n + x_{n-1}) + 1 = kx_n x_{n-1}. \tag{2.2c}$$

Starting with these expressions, it is very easy to construct the invariants for the ultradiscrete case. Let us work this out explicitly in the $\sigma = 2$ case. The ultradiscrete equation reads:

$$X_{n+1} + X_{n-1} + 2X_n = \max(X_n + A, 0) \tag{2.3}$$

and the ultradiscretization of the invariant (2.2c) leads to

$$K = \max(X_n + X_{n-1}, A - X_n, A - X_{n-1}, -X_n - X_{n-1}). \tag{2.4}$$

Let us now show that (2.4) is indeed a conserved quantity of (2.3). Starting from (2.4) and using $X_{n-1} = \max(X_n + A, 0) - 2X_n - X_{n+1}$, we obtain

$$\begin{aligned} &\max(X_n + X_{n-1}, A - X_n, A - X_{n-1}, -X_n - X_{n-1}) \\ &\quad = \max(\max(X_n + A, 0) - X_n - X_{n+1}, A - X_n, -\max(X_n + A, 0) \\ &\qquad + X_n + X_{n+1} + X_n + A, -\max(X_n + A, 0) + X_n + X_{n+1}) \\ &\quad = \max(\max(X_n + A, 0) - X_n - X_{n+1}, A - X_n, -\max(X_n + A, 0) \\ &\qquad + X_n + X_{n+1} + \max(X_n + A, 0)) \\ &\quad = \max(A - X_{n+1}, -X_n - X_{n+1}, A - X_n, X_n + X_{n+1}). \end{aligned} \tag{2.5}$$

This proves that K is indeed an invariant of (2.3).

Still, the existence of this invariant does not suffice in order to define the ultradiscrete analogue of an elliptic function. It is easy to see that the iteration of (2.4) considered as an ultradiscrete equation does not define X_{n+1} in terms of X_n in a unique way (in particular when the maximal term is $A - X_n$). Thus, we do not see how (2.4) can be used as such for the definition of the analogue of the elliptic functions. One must use the full three-point mapping (or find a different approach to this question). Let us first show a typical solution of (2.3), see figure 1. The periodicity that characterizes the elliptic functions is clearly seen in the solution.

Since the invariant, considered as a ultradiscrete equation, cannot be used to define the ultradiscrete elliptic functions a more direct approach is needed. We must, as usually, go back to the discrete case [6] and once the result is firmly established we can proceed to its ultradiscretization. Let us start with the differential equation

$$\frac{\mathrm{d}x}{\sqrt{P}} + \frac{\mathrm{d}y}{\sqrt{Q}} = 0 \tag{2.6}$$

where

$$P \equiv -(x - a)(x - b)(x - c) \qquad Q \equiv -(y - a)(y - b)(y - c)$$

and a, b, c are constants (complex in general). A standard argument gives an integral of the differential equation (2.6) as

$$\frac{(\sqrt{P} - \sqrt{Q})^2}{(x - y)^2} = -(x + y) + C \tag{2.7}$$

where C is an integration constant. If we choose $C = a + b + c$, equation (2.7) is equivalent to the following algebraic equation:

$$x^2 y^2 - 2\beta xy + \alpha(x + y) + \gamma = 0 \tag{2.8}$$

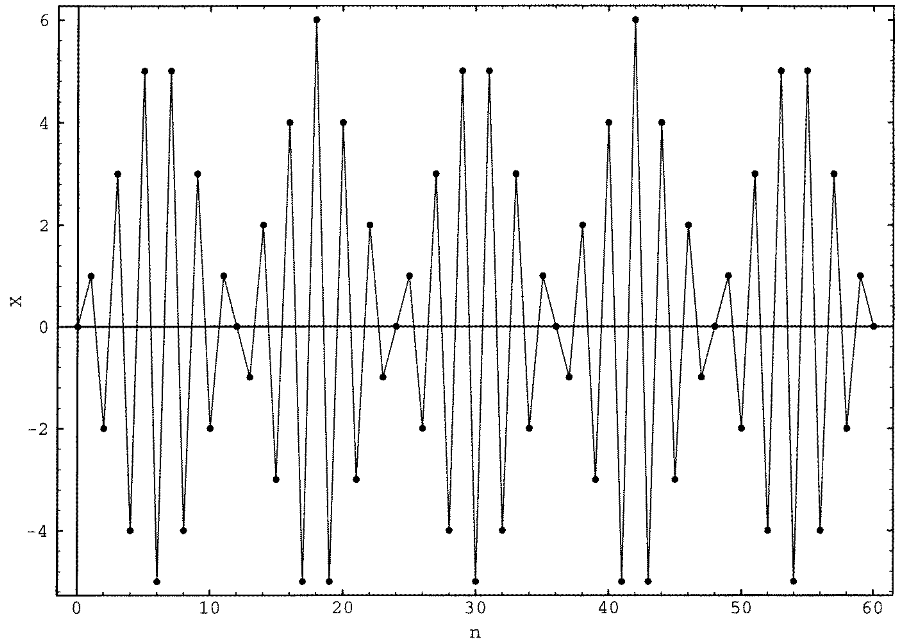

Figure 1. A typical solution of the autonomous ultradiscrete equation (2.3) corresponding to $X_0 = 0$, $X_1 = 1$, $A = -4$, with period 24.

where $\alpha \equiv 4abc$, $\beta \equiv ab + bc + ca$ and $\gamma \equiv a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)$. This is exactly the integral we need for the autonomous d-$\mathrm{P_I}$. In fact, if both $(x, y) = (x_{n+1}, x_n)$ and $(x, y) = (x_n, x_{n-1})$ satisfy equation (2.8) and $x_{n+1} \neq x_{n-1}$, we can easily show

$$x_{n+1}x_n^2x_{n-1} = \alpha x_n + \gamma. \tag{2.9}$$

In order to take ultradiscrete limit (u-limit), we confine ourselves to the case where $x_n > 0$ (for $\forall n$), $\alpha > 0$ and $\gamma > 0$. Hence we assume

$$\begin{aligned} &a > b > c > 0 \\ &a > x > b \qquad a > y > b \\ &a > \frac{bc}{(\sqrt{b} - \sqrt{c})^2}. \end{aligned} \tag{2.10}$$

These are sufficient conditions to the positivity we required. (It may be possible that we can obtain another u-limit when we choose different conditions.) Then, since (x, y) in equation (2.8) are a solution to equation (2.6), they satisfy

$$\int_x^a \frac{\mathrm{d}x}{\sqrt{P}} + \int_y^a \frac{\mathrm{d}y}{\sqrt{Q}} = \text{constant}.$$

Using the identity

$$\begin{aligned} \int_x^a \frac{\mathrm{d}x}{\sqrt{P}} &= \frac{2}{\sqrt{a-c}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a-x}{a-b}}; \sqrt{\frac{a-b}{a-c}}\right) \\ &\equiv \frac{2}{\sqrt{a-c}}u \end{aligned} \tag{2.11}$$

we have a parametrization

$$\begin{aligned} x &= a - (a-b)\,\mathrm{sn}^2(u;k) \\ y &= a - (a-b)\,\mathrm{sn}^2(v;k) \end{aligned} \tag{2.12}$$

with $u+v=\xi$ (constant) and $k \equiv \sqrt{\frac{a-b}{a-c}}$. (Here $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$ are Jacobian elliptic functions and k is the modulus.)

The constant ξ is obtained as follows. Differentiating equations (2.11) and (2.12), we find

$$\begin{aligned} -\frac{1}{\sqrt{P}} &= \frac{2}{\sqrt{a-c}}\frac{\mathrm{d}u}{\mathrm{d}x} \\ \frac{\mathrm{d}x}{\mathrm{d}u} &= -2(a-b)\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u. \end{aligned}$$

Thus we obtain

$$\sqrt{P} = \sqrt{a-c}(a-b)\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u. \tag{2.13}$$

Putting equations (2.12) and (2.13) into equation (2.7) with $C = a+b+c$, we obtain

$$\begin{aligned} \left(\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u - \mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v}{\mathrm{sn}^2\,v - \mathrm{sn}^2\,u}\right)^2 &= \frac{a-b}{a-c}(\mathrm{sn}^2\,u + \mathrm{sn}^2\,v) + \frac{b+c-a}{a-c} \\ &= k^2(\mathrm{sn}^2\,u + \mathrm{sn}^2\,v) - 1 + \frac{b}{a-c}. \end{aligned}$$

Since we know $u+v=\xi$ (constant), taking $u=0$ and $v=\xi$, we obtain

$$\frac{\mathrm{dn}^2\,\xi}{\mathrm{sn}^2\,\xi} = \frac{b}{a-c}$$

or equivalently,

$$\mathrm{sn}^2\,\xi = \mathrm{sn}^2(u+v) = \frac{a-c}{a}. \tag{2.14}$$

The above relation is also obtained from the identity

$$\left(\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u - \mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v}{\mathrm{sn}^2\,v - \mathrm{sn}^2\,u}\right)^2 - k^2(\mathrm{sn}^2\,u + \mathrm{sn}^2\,v) + (1+k^2) = \frac{1}{\mathrm{sn}^2(u+v)}$$

which can be proved by the addition formulae of Jacobian elliptic functions. We should regard equation (2.14) as the definition of ξ.

Thus we obtain an elliptic solution to equation (2.9). It is given as

$$x_n = f(u_0 - n\xi) \tag{2.15}$$

where

$$f(u) \equiv a - (a-b)\,\mathrm{sn}^2(u;k) = a\,\mathrm{cn}^2(u;k) + b\,\mathrm{sn}^2(u;k)$$

and we use the fact $f(-u) = f(u)$.

The ultradiscretization of the autonomous d-$\mathrm{P_I}$ can be done in a straightforward way. To begin with, we rewrite sn and cn functions in terms of elliptic theta functions. The definitons of elliptic theta functions are

$$\vartheta_0(\nu) \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^{2n}$$

$$\vartheta_1(\nu) \equiv \sqrt{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2} z^{2n-1}$$
$$\vartheta_2(\nu) \equiv \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} z^{2n-1}$$
$$\vartheta_3(\nu) \equiv \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n}$$

where $z = \exp[\sqrt{-1}\pi\nu]$ and q is a complex constant (nome). We set

$$q = \exp\left[-\frac{\epsilon\pi}{\theta}\right]. \tag{2.16}$$

Using the Poisson's summation formulae, we get

$$\vartheta_0(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\theta}{\epsilon}[\nu - (n + \tfrac{1}{2})]^2\right]$$
$$\vartheta_1(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{\theta}{\epsilon}[\nu - (n + \tfrac{1}{2})]^2\right]$$
$$\vartheta_2(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{\theta}{\epsilon}[\nu - n]^2\right]$$
$$\vartheta_3(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\theta}{\epsilon}[\nu - n]^2\right].$$

The relations between Jacobian sn, cn functions and theta functions are given,

$$\mathrm{sn}\, u = \frac{\vartheta_3(0)\vartheta_1(\nu)}{\vartheta_2(0)\vartheta_0(\nu)} \qquad \mathrm{cn}\, u = \frac{\vartheta_0(0)\vartheta_2(\nu)}{\vartheta_2(0)\vartheta_0(\nu)}$$

with $u = \pi(\vartheta_3(0))^2\nu \equiv K\nu$ and $k^2 = \frac{a-b}{a-c} = (\frac{\vartheta_2(0)}{\vartheta_3(0)})^4$.

We parametrize a, b, c as

$$c = \exp\left[-\frac{\eta\theta}{\epsilon}\right]$$
$$b = c + \left(\frac{\epsilon\pi}{\theta}\right)^2 (\vartheta_0(0))^4$$
$$a = c + \left(\frac{\epsilon\pi}{\theta}\right)^2 (\vartheta_3(0))^4.$$

(This parametrization with (2.16) is not a unique one. There may be other parametrizations which lead to different u-limits.) Noting the facts

$$\vartheta_0(0) \sim 2\sqrt{\frac{\theta}{\epsilon\pi}} \exp\left[-\frac{\theta}{4\epsilon}\right] \qquad (\epsilon \to +0)$$
$$\vartheta_2(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 - 2\exp\left[-\frac{\theta}{\epsilon}\right]\right) \qquad (\epsilon \to +0)$$
$$\vartheta_3(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right) \qquad (\epsilon \to +0)$$
$$(\vartheta_3(\nu))^4 = (\vartheta_0(\nu))^4 + (\vartheta_2(\nu))^4 - (\vartheta_1(\nu))^4$$

we find in the limit $\epsilon \to +0$,

$$a \sim 1 \qquad b \sim \exp\left[-\frac{\eta\theta}{\epsilon}\right] \qquad c \sim \exp\left[-\frac{\eta\theta}{\epsilon}\right] \qquad b - c \sim \exp\left[-\frac{\theta}{\epsilon}\right]$$

$$\alpha = 4abc \sim \exp\left[-\frac{2\eta\theta}{\epsilon}\right]$$

$$\gamma = (b-c)^2 a^2 + \cdots \sim \exp\left[-\frac{2\theta}{\epsilon}\right].$$

Since a, b, c must satisfy the inequality (2.10) and

$$\frac{bc}{(\sqrt{b}-\sqrt{c})^2} = \frac{bc(\sqrt{b}+\sqrt{c})^2}{(b-c)^2} \sim \exp\left[-\frac{(3\eta-2)\theta}{\epsilon}\right] \qquad (\epsilon \to +0)$$

we find the region of η as

$$\tfrac{2}{3} < \eta < 1.$$

We also find

$$(\vartheta_0(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu)) - \tfrac{1}{2}]^2\right] + \exp\left[-\frac{\theta}{\epsilon}[((\nu)) + \tfrac{1}{2}]^2\right]\right)^2$$

$$(\vartheta_1(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu)) - \tfrac{1}{2}]^2\right] - \exp\left[-\frac{\theta}{\epsilon}[((\nu)) + \tfrac{1}{2}]^2\right]\right)^2$$

$$(\vartheta_2(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu))]^2\right] - \exp\left[-\frac{\theta}{\epsilon}[((\nu)) - 1]^2\right]\right)^2$$

where $((\nu)) \equiv \nu - \mathrm{Floor}(\nu)$, and $\mathrm{Floor}(x)$ is the maximum integer which does not exceed x. Thus we get the asymptotic form of Jacobian elliptic functions as

$$\mathrm{sn}^2\, u \sim 1 \qquad \mathrm{cn}^2\, u \sim \left(\exp\left[-\frac{2\theta}{\epsilon}((\nu))\right] + \exp\left[-\frac{2\theta}{\epsilon}[1-((\nu))]\right]\right).$$

From equation (2.14) or $\mathrm{cn}^2(u+v) = \frac{c}{a}$, we also get a relation for $\nu \equiv \frac{1}{K}u$ and $\nu' \equiv \frac{1}{K}v$ in the limit $\epsilon \to +0$,

$$\eta = 2\min[((\nu+\nu')), 1 - ((\nu+\nu'))].$$

Thus we find

$$\nu' = \pm\frac{\eta}{2} - \nu + \text{an arbitrary integer}.$$

Using the identity

$$\min[x, y] = -\max[-x, -y] = \lim_{\epsilon\to+0} -\epsilon \log\left[\exp\left(-\frac{x}{\epsilon}\right) + \exp\left(-\frac{y}{\epsilon}\right)\right]$$

we obtain the u-limit of equation (2.9) as

$$X_{n+1} + 2X_n + X_{n-1} = \max[X_n + (\tfrac{3}{2} - 2\eta)\theta, 0] \tag{2.17}$$

where X is related to x through $x = \exp(\frac{X-\theta/2}{\epsilon})$. The elliptic solution of equation (2.17) is obtained from equation (2.15) using the asymptotic properties of elliptic functions as

$$X_n = \theta(\tfrac{1}{2} - \min[\eta, 2\nu_n, 2 - 2\nu_n]) \tag{2.18}$$

where

$$\nu_n = \left(\left(\nu_0 - n\frac{\eta}{2}\right)\right) = \nu_0 - n\frac{\eta}{2} - \mathrm{Floor}\left(\nu_0 - n\frac{\eta}{2}\right).$$

It should be noted that if $x_n(\epsilon)$ is a solution to equation (2.9) and the limit $\lim_{\epsilon\to+0}\log x_n(\epsilon) \equiv X_n - \theta/2$ exists, then X_n is a solution to equation (2.17). This fact proves that (2.18) is a solution to equation (2.17). (Note that equation (2.18) is invariant under exchange of $\nu_0 - n\frac{\eta}{2}$ by $-\nu_0 + n\frac{\eta}{2}$.)

If we set $\theta = p$, $\eta = \frac{q}{p}$ and $\nu_0 = \frac{r}{p}$ where p, q, r ($\frac{2}{3} < \frac{q}{p} < 1$) are integers, we can regard equation (2.17) as an evolution equation which takes values only in integers. It is also easy to see that X_n is periodic with period at most $2p$. Choosing θ, η and ν_0 we can show that (2.18) reproduces exactly the solution displayed in figure 1. Thus we have explicitly constructed the elliptic function solutions to the autonomous u-P_I (2.4). In a similar way we can proceed to the construction of the solutions of the other autonomous forms.

3. Special solutions of u-P_{II} and u-P_{III}

It is well known that the continuous and discrete Ps can be transformed into bilinear forms involving τ functions [7]. Each of the latter is an entire function associated with the singularity of the Ps, that is, the singularities are produced by the zeros of τ functions appearing in the denominator of the nonlinear variable of Ps. However, in the ultradiscrete case, the analogues of the notions of singularity and entire function have not yet been clearly defined. Then what is the τ function in the ultradiscrete world? In a naive sense, an entire function does not have any factor in the denominator. Therefore it is naturally expected that the τ function of ultradiscrete case has no term of negative sign and can be represented as a sum of only positive terms. (This is due to the fact that the u-limit '$\lim_{\epsilon\to+0}\epsilon\log$' transforms the factors in the numerator into the terms with positive sign and those in the denominator into the terms with negative sign.) Let us consider the simplest case, the rational solutions of Ps. For the continuous Ps, the τ functions of rational solutions are expressed by a Wronskian determinant with simple polynomial components. Also for discrete Ps, it has been shown (at least for the cases analysed up to now) that the τ functions of rational solutions are given in the form of a Casorati determinant, the discrete analogue of the Wronskian. In both cases, the bilinear equations of τs are directly obtained by using a simple determinant identity, the so-called Plücker relation [4]. We can thus expect that the u-Ps are transformed into an ultradiscrete bilinear form in terms of τ functions which are expressible as a termwise positive sum, and in that expression the derivation of the bilinear equations becomes transparent.

In the following, we give the rational solutions of u-P_{II} and u-P_{III} which are represented in a form reminiscent of the ultradiscrete analogues of the Casorati determinant.

Let us start with the u-P_{II} of first kind,

$$X_{n+1} + X_{n-1} = \max(0, n - X_n) - \max(0, X_n + n - a) \tag{3.1}$$

which is derived from the multiplicative d-P_{II},

$$x_{n+1}x_{n-1} = \frac{x_n + z}{x_n(1 + \alpha z x_n)} \qquad z = \lambda^n \qquad \alpha\text{: parameter} \tag{3.2}$$

by replacing $x = e^{X/\epsilon}$, $\lambda = e^{1/\epsilon}$, $\alpha = e^{-a/\epsilon}$ and taking the limit $\epsilon \to +0$. Through the variable transformation,

$$X_n = \tau_{n-1} - \tau_{n-2} \tag{3.3}$$

we obtain the ultradiscrete 'bilinear' equation,

$$\tau_n + \max(\tau_{n-2}, \tau_{n-1} + n - a) = \tau_{n-3} + \max(\tau_{n-1}, \tau_{n-2} + n). \tag{3.4}$$

We remark that in the ultradiscrete world, 'max' and '+' should be regarded as 'addition' and 'multiplication' respectively, because $\mathrm{e}^{A/\epsilon} + \mathrm{e}^{B/\epsilon}$ and $\mathrm{e}^{A/\epsilon}\mathrm{e}^{B/\epsilon}$ go to $\max(A, B)$ and $A+B$ respectively under the operation of $\lim_{\epsilon\to+0} \epsilon \log$. Thus the above equation is indeed the bilinear form in τ. Now (3.1) is invariant under the transformation $a \to -a$, $n \to n-a$, $X \to -X$, thus we can assume $a \geqslant 0$ without loss of generality. The u-$\mathrm{P_{II}}$ (3.1) admits rational solutions for $a = 4m$, m: non-negative integer. The τ function for the rational solution is given as

$$\tau_n = \sum_{j=0}^{m-1} \max(0, n - 3j) \tag{3.5}$$

which is also expressed as

$$\tau_n = \max_{0\leqslant j\leqslant m} (jn - \tfrac{3}{2} j(j-1)). \tag{3.6}$$

The existence of the above two expressions is essential in the proof of the bilinear equations as we will see below. This τ_n gives the m-step solutions X_n (3.3) which have m successive jumps of step 1 at $n = 3j - 1$, $1 \leqslant j \leqslant m$.

Let us consider a slightly more general form of the τ function,

$$\tau_n = \sum_{j=0}^{m-1} \max(0, n - jk) \tag{3.7a}$$

$$= \max_{0\leqslant j\leqslant m} \left(jn - \frac{j(j-1)}{2} k \right) \tag{3.7b}$$

where k is positive. For positive p and q, we have

$$\begin{aligned}\max(\tau_n, \tau_{n+p-k} + n - (m-1)p - q) = \max\Bigg(&\max_{0\leqslant j\leqslant m} \left(jn - \frac{j(j-1)}{2} k \right), \\ &\max_{1\leqslant j\leqslant m+1} \left(jn - \frac{j(j-1)}{2} k + (j-m)p - q \right) \Bigg)\end{aligned}$$

on the r.h.s. the jth term in the second max is less than the jth term in the first max for $1 \leqslant j \leqslant m$, thus we can drop these terms,

$$\begin{aligned} &= \max\left(\max_{0\leqslant j\leqslant m} \left(jn - \frac{j(j-1)}{2} k \right), (m+1)n - \frac{m(m+1)}{2} k + p - q \right) \\ &= \max\Bigg(0, n, 2n-k, \ldots, mn - \frac{m(m-1)}{2} k, \\ &\qquad (m+1)n - \frac{m(m+1)}{2} k + p - q \Bigg) \end{aligned}$$

and taking $p - q \leqslant k$, we obtain

$$\begin{aligned} &= \max(0, n) + \max(0, n-k) + \cdots + \max(0, n-(m-1)k) \\ &\quad + \max(0, n - mk + p - q) \end{aligned}$$

because the second arguments of the above maxima are numbers in decreasing order. Hence we obtain

$$\begin{aligned} &\max(\tau_n, \tau_{n+p-k} + n - (m-1)p - q) = \tau_n + \max(0, n - mk + p - q) \\ &\qquad p \geqslant 0, \;\; q \geqslant 0, \;\; p - q \leqslant k. \end{aligned} \tag{3.8}$$

Similarly we have

$$\max(\tau_n, \tau_{n+p-k}+n+q) = \max\left(\max_{0\leqslant j\leqslant m}\left(jn-\frac{j(j-1)}{2}k\right),\right.$$
$$\left.\max_{1\leqslant j\leqslant m+1}\left(jn-\frac{j(j-1)}{2}k+(j-1)p+q\right)\right)$$

where the jth term in the second max is dominant for $1\leqslant j\leqslant m$, so

$$= \max\left(0, \max_{1\leqslant j\leqslant m+1}\left(jn-\frac{j(j-1)}{2}k+(j-1)p+q\right)\right)$$
$$= \max\left(0, n+q, 2n+q+p-k, \ldots, (m+1)n+q+mp-\frac{m(m+1)}{2}k\right)$$

again taking $p-q\leqslant k$, we obtain

$$= \max(0, n+q)+\max(0, n+p-k)+\max(0, n+p-2k)+\cdots$$
$$+\max(0, n+p-mk).$$

Thus we obtain

$$\max(\tau_n, \tau_{n+p-k}+n+q) = \max(0, n+q)+\tau_{n+p-k} \qquad p\geqslant 0,\ q\geqslant 0,\ p-q\leqslant k. \tag{3.9}$$

From (3.8) through replacing $p\to p+q$, $n\to n-p$, and (3.9) through replacing $n\to n-r$, $p\to k+r-p$, $q\to r$, we get the bilinear equation,

$$\tau_n+\max(\tau_{n-p}, \tau_{n+q-k}+n-m(p+q)) = \tau_{n-k}+\max(\tau_{n-r}, \tau_{n-p}+n)$$
$$0\leqslant p\leqslant k,\ q\geqslant 0,\ r\geqslant 0 \tag{3.10}$$

where we have used $\tau_n-\tau_{n-k} = \max(0, n)-\max(0, n-mk)$ which follows from the explicit expression of τ (3.7a).

For $k=3$, $p=2$, $q=2$ and $r=1$, (3.10) reduces to (3.4), thus we have proved that (3.3) and (3.5) give the solution of u-P_{II} (3.1) with $a=4m$. The remark is that expression (3.7a) looks similar to a Casorati determinant and (3.7b) to the expansion of the determinant. Let us recall the Casorati determinant representation of m-soliton solution for the discrete KP hierarchy [8]. The determinant τ consists of the products of m components and each component of the determinant is the sum of two terms. Expanding the determinant, we get the expression of the sum of exponential terms. This situation is parallel to (3.7a) and (3.7b) except for a missing 'max' in front of the summation in (3.7a).

Next we will consider the special solution for the u-P_{II} of second kind,

$$X_{n+1}+X_{n-1}-X_n = \max(0, n-X_n)-\max(0, X_n+n-a). \tag{3.11}$$

This is derived from another multiplicative d-P_{II},

$$x_{n+1}x_{n-1} = \frac{x_n+z}{1+\alpha z x_n} \qquad z=\lambda^n \qquad \alpha\text{: parameter} \tag{3.12}$$

through $x=e^{X/\epsilon}$, $\lambda=e^{1/\epsilon}$, $\alpha=e^{-a/\epsilon}$ and $\epsilon\to 0$. The ultradiscrete bilinear form of (3.11) is

$$\tau_n+\max(\tau_{n-3}, \tau_{n-1}+n-a) = \tau_{n-4}+\max(\tau_{n-1}, \tau_{n-3}+n) \tag{3.13}$$

where $X_n=\tau_{n-1}-\tau_{n-3}$. For $a=6m$, m: integer, there exist rational solutions of (3.12). The τ function for the solution is given by

$$\tau_n = \sum_{j=0}^{m-1}\max(0, n-4j) \tag{3.14}$$

and the bilinear equation (3.13) is just the consequence of (3.10) with $k = 4$, $p = 3$, $q = 3$ and $r = 1$. For this τ function, X_n gives a multistep solution with the elementary pattern of two successive jumps at $n = 4j - 2$ $(1 \leqslant j \leqslant m)$ followed by two steps with constant value.

Let us proceed to the rational solution of u-P_{III}. We consider only a degenerate case, namely, the case in which the u-P_{III} is decomposed into two parts. The u-P_{III},

$$X_{n+1} + X_{n-1} - 2X_n = \max(0, n - X_n) - \max(0, X_n + n - a) + \max(0, n - X_n + b) - \max(0, X_n + n + b) \tag{3.15}$$

is derived from the d-P_{III},

$$x_{n+1}x_{n-1} = \frac{(x_n + z)(x_n + \beta z)}{(1 + \alpha z x_n)(1 + \beta z x_n)} \qquad z = \lambda^n \qquad \alpha, \beta\text{: parameter} \tag{3.16}$$

through $x = e^{X/\epsilon}$, $\lambda = e^{1/\epsilon}$, $\alpha = e^{-a/\epsilon}$, $\beta = e^{b/\epsilon}$ and $\epsilon \to +0$. Now decomposing (3.15) in the following two equations,

$$X_{n+1} + X_{n-1} = \max(0, n - X_n) - \max(0, X_n + n - a) \tag{3.17a}$$

$$2X_n = \max(0, X_n + n + b) - \max(0, n - X_n + b) \tag{3.17b}$$

we get the bilinear equations through $X_n = \tau_{n-1} - \tau_{n-2}$,

$$\tau_n + \max(\tau_{n-2}, \tau_{n-1} + n - a) = \tau_{n-3} + \max(\tau_{n-1}, \tau_{n-2} + n) \tag{3.18a}$$

$$\tau_{n-1} + \max(\tau_{n-1}, \tau_{n-2} + n + b) = \tau_{n-2} + \max(\tau_{n-2}, \tau_{n-1} + n + b). \tag{3.18b}$$

The first equation (3.17*a*) or (3.18*a*) is nothing but the u-P_{II} of the first kind, therefore we have to prove that the solution (3.5) simultaneously satisfies the second bilinear equation (3.18*b*). From (3.9) we obtain another bilinear equation,

$$\max(\tau_{n-q_1}, \tau_{n-p_1} + n) - \tau_{n-p_1} = \max(\tau_{n-q_2}, \tau_{n-p_2} + n) - \tau_{n-p_2} \qquad 0 \leqslant p_i \leqslant q_i + k, \ q_i \geqslant 0 \tag{3.19}$$

which gives (3.18*b*) by taking $p_1 = q_2 = b + 1$, $p_2 = q_1 = b + 2$ and $k = 3$. Hence we have proved that the same τ as in the rational solution of u-P_{II} gives the solution for the u-P_{III} (3.15) with $a = 4m$ (m: integer) and $b \geqslant -1$.

Solutions to the higher u-Ps could be obtained following the above techniques (obviously with considerable technical difficulties).

4. Solutions of the P_I equation: continuous, discrete and ultradiscrete cases

In the previous sections, we have shown that, in perfect parallel to the continuous and discrete cases, the u-P_{II} and u-P_{III} possess special solutions. Thus, naturally, the question arises of how does the solution of the ultradiscrete equation relate to the solution of its continuous homologue. In order to perform this comparison we have chosen to analyse the P_I equation. The reason is that this equation does not have any special solutions and thus an arbitrary choice of initial conditions is expected to yield the generic behaviour of the solution. Our argument is (and has always been) that the *discrete* equation captures the essence of the behaviour of both its limiting cases, be it continuous or ultradiscrete. In order to show this explicitly we shall consider a specific example. We start from the discrete form of P_I: $x_{n-1}x_{n+1} = zx_n + 1$ where $z = a\lambda^n$. We introduce a scaling of x and z so as to rewrite the equation as

$$x_{n-1}x_{n+1} = zx_n + b. \tag{4.1}$$

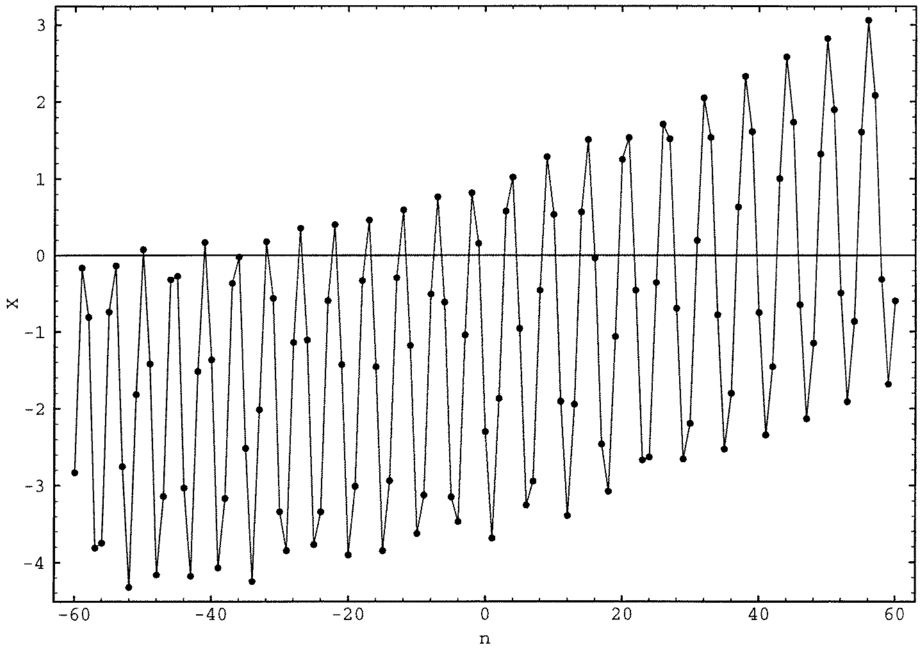

Figure 2. Solution of the d-P_I (4.1) corresponding to $x_0 = 0.1$, $x_1 = 0.025$, $\lambda = 1.01$, $a = 0.2$ and $b = +0.01$. The behaviour is reminiscent of the one of the 'automaton' equation (4.2).

We can, without loss of generality, assume that $a > 0$ (If $a < 0$ it suffices to change the sign of both x and z). Let us first look at the ultradiscrete case. For this we must assume that $b > 0$, and choosing initial conditions $x_{n-1}, x_n > 0$ we find $x_{n+1} > 0$. Thus we can take the logarithm of x and introduce the new variable $X = \epsilon \log x$ where $\epsilon = 1/\log \lambda$. Figure 2 shows a typical behaviour of the solution of (4.1) for positive b where we have plotted the variable $X = \log x_n$ as a function of n.

Clearly this behaviour is only vaguely reminiscent of that of the continuous P_I: only the growing oscillating part for positive n resembles the one of the solution of the latter. The double poles present in the solution of P_I are absent.

The u-limit corresponds to $\lambda \to \infty$ (or $\epsilon \to 0$) and leads to the equation

$$X_{n-1} + X_{n+1} = \max(X_n + n + A, B). \tag{4.2}$$

Going to the limit does not change the overall appearence of the solution of (4.2) as compared with that of (4.1) presented in figure 2. Simply, the values of X are now integers provided we start with integer values for X_{n-1}, X_n, A and B.

In order to obtain the continuous limit of (4.1), we introduce the following transformations: $x = \beta(1 + \epsilon^2 w)$ and $z = 2\beta(1 + \epsilon^4 t)$ with $b = -\beta^2$ and $t = n\epsilon$. The essential observation here is that for the continuous limit to exist we must have $b < 0$. In the limit $\epsilon \to 0$, we obtain the continuous P_I in the form

$$w'' + w^2 - 2t = 0. \tag{4.3}$$

In figure 3 we plot the solution of (4.1) for $b < 0$ where we represent $X = x/\beta - 1$ as a function of n. The appearance of poles is now clear.

These two simulations confirm our statement that the solution of the discrete equation contains the full richness of behaviour. In the particular example studied here, depending

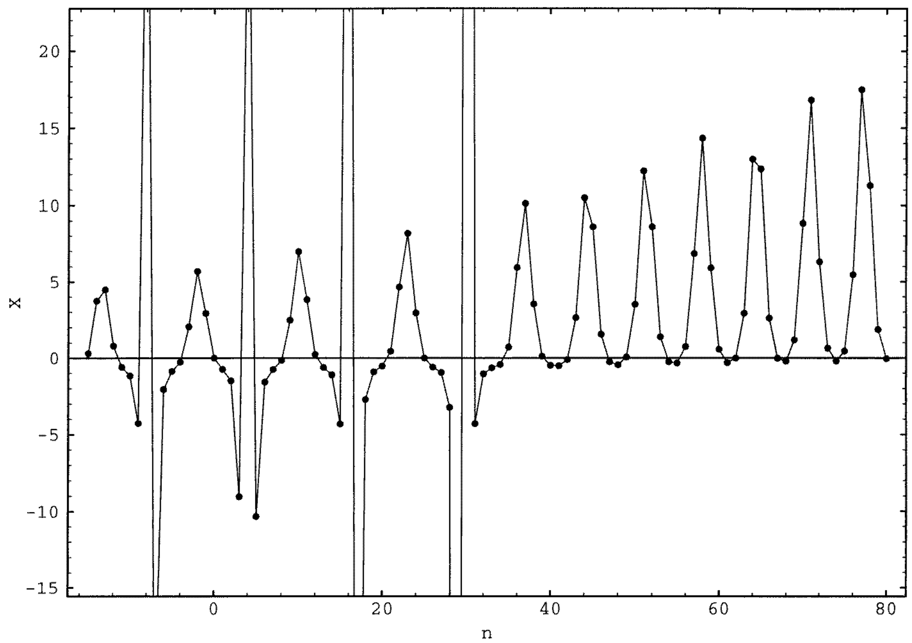

Figure 3. Solution of the d-P$_{\mathrm{I}}$ (4.1) corresponding to $x_0 = 0.1$, $x_1 = 0.025$, $\lambda = 1.04$, $a = 0.135$ and $b = -0.01$. Note the appearance of poles for large negative ns.

on the sign of b, we find ourselves either in the ultradiscrete or the continuous domain of behaviour. The precise limits are not essential: as soon as the sign of b changes, the qualitative changes in the solutions set in.

Finally, let us for the sake of completeness study the case $b = 0$. In this case (4.1) becomes

$$x_{n-1}x_{n+1} = zx_n \tag{4.4}$$

and introducing again $X = \epsilon \log x$ we have the linear equation

$$X_{n-1} + X_{n+1} = X_n + n. \tag{4.5}$$

The solutions of (4.5) is straightforward: $X = n + \mu j^n + \nu j^{2n}$ where $j = \mathrm{e}^{\mathrm{i}\pi/3}$, i.e. a linear growth with a superimposed oscillating pattern of period 6.

5. Conclusion

In the preceding sections, we have studied the special solutions of the u-Ps. We have shown that the analogue of the bilinear formalism exists also in the ultradiscrete case and that it can be used in order to construct the ultradiscrete analogue of the Casorati determinant solutions of the Ps. This work is a first exploratory investigation in this direction: a complete study of the special solutions of the u-Ps will necessitate a considerable effort (as a matter of fact, this programme is not yet complete even in the continuous and discrete cases).

The autonomous limit of the u-Ps has also been considered. In analogy to the continuous and discrete cases we expected the equations to possess an explicit invariant and be solved in terms of elliptic functions. We have shown that these statements are true and we have explicitly constructed the ultradiscrete analogue of elliptic functions.

Finally our work has (hopefully) helped to illustrate the fundamental character of the discrete equations. Choosing the specific example of P_I, we have shown by detailed numerical simulations that the discrete P_I can exhibit behaviours reminiscent of both the continuous and u-P_I depending on the sign of a parameter. This is a further indication that the discrete equations are the fundamental entities of the integrable world.

Acknowledgments

BG and AR are deeply indebted to J Satsuma for his invitation to the Graduate School of Mathematical Sciences of the University of Tokyo that made possible the completion of this collaboration. The help of O Gérard with the numerical simulations is also gratefully acknowledged. YO acknowledges the financial support of the Japan Ministry of Education through the Foreign Study Program.

References

[1] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247
[2] Grammaticos B, Ohta Y, Ramani A, Takahashi D and Tamizhmani K M 1997 *Phys. Lett.* A **226** 53
[3] Ramani A and Grammaticos B 1996 *Physica* **228A** 160
[4] Kajiwara K, Ohta Y, Satsuma J, Grammaticos B and Ramani A 1994 *J. Phys. A: Math. Gen.* **27** 915
Kajiwara K, Ohta Y and Satsuma J 1995 *J. Math. Phys.* **36** 4162
Kajiwara K and Ohta Y 1996 *J. Math. Phys.* **37** 4693
[5] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* **34D** 183
[6] Toda M 1976 *Da-en Kansu Nyumon (Introduction to Elliptic Functions)* (Tokyo: Nihon-Hyoron-sha) (in Japanese)
[7] Ohta Y, Ramani A, Grammaticos B and Tamizhmani K M 1996 *Phys. Lett.* A **216** 255
Ramani A, Grammaticos B and Satsuma J 1995 *J. Phys. A: Math. Gen.* **28** 4655
Hietarinta J and Kruskal M D 1992 *Hirota Forms for the Six Painlevé Equations from Singularity Analysis (NATO ASI Series B278)* (New York: Plenum) p 175
[8] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943
Ohta Y, Hirota R, Tsujimoto S and Imai T 1993 *J. Phys. Soc. Japan* **62** 1872